

## Euclidean Spaces

A vector space  $E$  is called a Euclidean Space if it has a SCALAR PRODUCT defined in it.

$$E \times E \longrightarrow \mathbb{R}$$

$$\langle \bar{u}, \bar{v} \rangle = \mu \in \mathbb{R}$$

Scalar Product in  $E$

SYMMETRY  $\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$

LINEALITY  $\langle \alpha \bar{u} + \beta \bar{v}, \bar{w} \rangle = \alpha \langle \bar{u}, \bar{w} \rangle + \beta \langle \bar{v}, \bar{w} \rangle$

POSITIVITY  $\langle \bar{u}, \bar{u} \rangle > 0$

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Example: Prove  $\forall p(x), q(x) \in \mathbb{P}_2$   $\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x) dx$  is a scalar product of  $\mathbb{P}_2$ .

① Symmetry  $\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x) dx = \int_0^1 q(x)p(x) dx = \langle q(x), p(x) \rangle$  ✓

② Linearity  $\langle \alpha p(x) + \beta q(x), t(x) \rangle = \int_0^1 (\alpha p(x) + \beta q(x))t(x) dx = \int_0^1 (\alpha p(x)t(x) + \beta q(x)t(x)) dx =$

$= \alpha \int_0^1 p(x)t(x) dx + \beta \int_0^1 q(x)t(x) dx = \alpha \langle p(x), t(x) \rangle + \beta \langle q(x), t(x) \rangle$  ✓

③ Positivity  $\langle p(x), p(x) \rangle = \int_0^1 p(x)p(x) dx = \int_0^1 \underbrace{[p(x)]^2}_{>0} dx > 0$  ✓

Yes, it is a Scalar Product

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# Gramm's matrix

In  $\mathbb{R}^3(\mathbb{R})$  with a base  $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$$\langle \bar{u}, \bar{v} \rangle = \underbrace{\mu}_{\in \mathbb{R}} \iff \underbrace{(u^1 \ u^2 \ u^3)}_{\bar{u}} \underbrace{\begin{pmatrix} \langle \bar{e}_1, \bar{e}_1 \rangle & \langle \bar{e}_1, \bar{e}_2 \rangle & \langle \bar{e}_1, \bar{e}_3 \rangle \\ \langle \bar{e}_1, \bar{e}_2 \rangle & \langle \bar{e}_2, \bar{e}_2 \rangle & \langle \bar{e}_2, \bar{e}_3 \rangle \\ \langle \bar{e}_1, \bar{e}_3 \rangle & \langle \bar{e}_2, \bar{e}_3 \rangle & \langle \bar{e}_3, \bar{e}_3 \rangle \end{pmatrix}}_{G_B \equiv \text{Gramm's Matrix}} \underbrace{\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}}_{\bar{v}} = \underbrace{\mu}_{\in \mathbb{R}}$$

e.g. Build the Gramm Matrix of the previous example for base  $B = \{x^2, x, 1\}$

$$G_B = \begin{pmatrix} \langle x^2, x^2 \rangle & \langle x^2, x \rangle & \langle x^2, 1 \rangle \\ \langle x^2, x \rangle & \langle x, x \rangle & \langle x, 1 \rangle \\ \langle x^2, 1 \rangle & \langle x, 1 \rangle & \langle 1, 1 \rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 \end{pmatrix}$$

$$\begin{aligned} \langle x^2, x^2 \rangle &= \int_0^1 x^2 x^2 dx = \frac{1}{5} & \langle x, x \rangle &= \frac{1}{3} \\ \langle x^2, x \rangle &= \int_0^1 x^2 x dx = \frac{1}{4} & \langle x, 1 \rangle &= \frac{1}{2} \\ \langle x^2, 1 \rangle &= \int_0^1 x^2 \cdot 1 dx = \frac{1}{3} & \langle 1, 1 \rangle &= 1 \end{aligned}$$

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ORTHOGONALITY  $\bar{u}$  and  $\bar{v}$  are ORTOGONAL  $\Leftrightarrow \langle \bar{u}, \bar{v} \rangle = 0$

MODULUS  $|\bar{u}| = \sqrt{\langle \bar{u}, \bar{u} \rangle}$  Modulus of  $\bar{u}$

ANGLES  $\langle \bar{u}, \bar{v} \rangle = |\bar{u}| |\bar{v}| \cos(\widehat{\bar{u}, \bar{v}}) \longrightarrow (\widehat{\bar{u}, \bar{v}}) = \arccos \frac{\langle \bar{u}, \bar{v} \rangle}{|\bar{u}| |\bar{v}|} \equiv \text{angle formed by } \bar{u} \text{ and } \bar{v}$

Orthogonal Subspace ( $\omega(L)$  or  $L^\perp$ )

Given  $L$ , a subspace of  $E$ ,  $\omega(L)$  is the orthogonal subspace to  $L$ , where:

$$\omega(L) = \{ \bar{x} \in E / \langle \bar{x}, \bar{y} \rangle = 0 \quad \forall \bar{y} \in L \}$$

Orthogonality and Gramm

A vector is NORMAL when  $|\bar{u}| = 1$

$$\begin{pmatrix} \langle \bar{e}_1, \bar{e}_1 \rangle & \langle \bar{e}_1, \bar{e}_2 \rangle & \langle \bar{e}_1, \bar{e}_3 \rangle \\ \langle \bar{e}_1, \bar{e}_2 \rangle & \langle \bar{e}_2, \bar{e}_2 \rangle & \langle \bar{e}_2, \bar{e}_3 \rangle \\ \langle \bar{e}_1, \bar{e}_3 \rangle & \langle \bar{e}_2, \bar{e}_3 \rangle & \langle \bar{e}_3, \bar{e}_3 \rangle \end{pmatrix}$$

$B$  is an ORTHOGONAL base  $\Leftrightarrow \begin{cases} \langle \bar{e}_1, \bar{e}_2 \rangle = 0 \\ \langle \bar{e}_1, \bar{e}_3 \rangle = 0 \\ \langle \bar{e}_2, \bar{e}_3 \rangle = 0 \end{cases} \Leftrightarrow G_B = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$

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$$|\bar{e}_1| = |\bar{e}_2| = |\bar{e}_3| = 1$$

## The usual scalar product (orthonormal base)

$$\bar{x} = (x^1, x^2, x^3)_B$$

$$\bar{y} = (y^1, y^2, y^3)_B$$

$B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$  an ORTHONORMAL  
 ↗ Base of  $\mathbb{R}^3$

$$\begin{aligned} \langle \bar{e}_1, \bar{e}_2 \rangle &= 0 & |\bar{e}_1| &= 1 \\ \langle \bar{e}_1, \bar{e}_3 \rangle &= 0 & |\bar{e}_2| &= 1 \\ \langle \bar{e}_2, \bar{e}_3 \rangle &= 0 & |\bar{e}_3| &= 1 \end{aligned}$$

$$\Leftrightarrow G_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\begin{aligned} \langle \bar{x}, \bar{y} \rangle &= \overbrace{(x^1 \ x^2 \ x^3)}_{\bar{x}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{G_B} \overbrace{\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}}_{\bar{y}} = \\ &= x^1 y^1 + x^2 y^2 + x^3 y^3 \end{aligned}$$

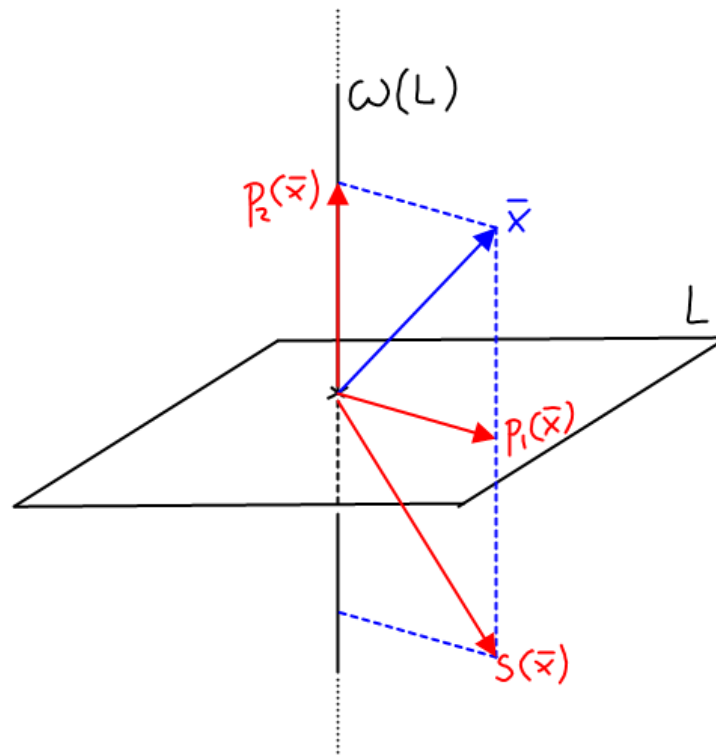
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Projection of a vector  $\bar{x}$  on  $L$  and  $\omega(L)$



Since  $\omega(L) \oplus L$  ALWAYS

$$\forall \bar{x} \in E \quad \bar{x} = P_1(\bar{x}) + P_2(\bar{x})$$

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